Exam

25/01/2024, 8:30 am - 10:30 am

Instructions:

- Prepare your solutions in an **ordered, clear and clean way**. Avoid delivering solutions with scratches.
- Write your name and student number in **all** pages of your solutions.
- Clearly indicate each exercise and the corresponding answer. Provide your solutions with as much detail as possible.
- Use different pieces of paper for solutions of different exercises.
- Read first the whole exam, and make a strategy for which exercises you attempt first. Start with those you feel comfortable with! There are $8+3$ total exercises (3 bonus exercises).

Exercise 1: (0.5+0.5 points) Consider the function on $\mathbb{R}^3$ defined by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{cases} \dfrac{xyz}{x^4+y^4+z^4} & \text{if } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ 0 & \text{if } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{cases}$$

a) Show that all partial derivatives exist everywhere.

b) Where is f differentiable?

Solution:

a) We have

$$\begin{aligned} \mathrm{D}_1 f = \frac{\partial f}{\partial x} &= \frac{yz}{x^4+y^4+z^4} - \frac{4x^4yz}{(x^4+y^4+z^4)^2} \\ \mathrm{D}_2 f = \frac{\partial f}{\partial y} &= \frac{xz}{x^4+y^4+z^4} - \frac{4xy^4z}{(x^4+y^4+z^4)^2} \\ \mathrm{D}_3 f = \frac{\partial f}{\partial z} &= \frac{xy}{x^4+y^4+z^4} - \frac{4xyz^4}{(x^4+y^4+z^4)^2} \end{aligned} \tag{1}$$

The only place where the partial derivatives may not be defined is at the origin. However, note that f and $\mathrm{D}_i f$, $(i=1,2,3)$, vanish along the axes, hence each partial derivative is 0 at the origin and so the derivatives exist everywhere . Recall also corollary 1.8.2 from the book.

b) According to the theorem of slide 15 of lecture 1, the given function is differentiable in $\mathbb{R}^3 \setminus \{0\}$ since $f$is not even continuous at 0.

Exercise 2: (1.5 points) Let $U \subset \mathrm{Mat}(n,n)$ be the set of matrices A such that the matrix $AA^\top + A^\top A$ is invertible. Compute the derivative of the map $F: U \to \mathrm{Mat}(n,n)$ given by

$$F(A) = \left(AA^\top + A^\top A\right)^{-1}.$$

Solution: Let $g(A) = AA^\top + A^\top A$ and $f(A) = A^{-1}$. Then $F(A) = f \circ g(A)$. From exercise 5 of tutorial 1 we know that $\mathrm{D}(AA^\top) = AH^\top + HA^\top$, therefore $\mathrm{D}g(A) = AH^\top + HA^\top + A^\top H + H^\top A$, while from lecture 1 we know that $\mathrm{D}f(A) = -A^{-1}HA^{-1}$. Using the chain rule (slide 12 of lecture 1) we have that

$$\mathrm{D}F(A) = -(AA^\top + A^\top A)^{-1}(AH^\top + HA^\top + A^\top H + H^\top A)(AA^\top + A^\top A)^{-1}.$$

Exercise 3: (1 point) Consider the mapping $S : \mathrm{Mat}(2,2) \to \mathrm{Mat}(2,2)$ given by $S(A) = A^2$. Observe that $S(-I) = I$. Does there exist an inverse mapping g, i.e., a mapping such that $S(g(A)) = A$, defined in a neighborhood of I, and such that $g(I) = -I$?

Solution: The answer is yes, we just need to confirm the conditions of the inverse function theorem. Recall that we know that $\mathrm{D}S(A) : H \mapsto AH + HA$. This is a linear function, hence continuous, and it is clearly invertible at $A = -I$. Therefore, by the inverse function theorem (slide 9 of lecture 3), the function S is invertible in a neighborhood of $-I$ and there exists a C^1-mapping g such that $g(I) = -I$ and $S(g(A)) = A$ for A in a neighborhood of I.

Exercise 4: (1 point) Show that if $f\begin{pmatrix} x \\ y \end{pmatrix} = \varphi(x-y)$ for some twice continuously differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$, then $\mathrm{D}_1^2 f - \mathrm{D}_2^2 f = 0$

Solution: This is straightforward by using the chain rule. Let $z = x - y$. Then one confirms that indeed $\mathrm{D}_1^2 f = \mathrm{D}_2^2 f$, leading to the result:

$$\begin{aligned} \mathrm{D}_1 f &= \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial\varphi}{\partial z} \\ \mathrm{D}_1^2 f &= \frac{\partial^2\varphi}{\partial z^2}\frac{\partial z}{\partial x} = \frac{\partial^2\varphi}{\partial z^2} \\ \mathrm{D}_2 f &= \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial y} = -\frac{\partial\varphi}{\partial z} \\ \mathrm{D}_2^2 f &= -\frac{\partial^2\varphi}{\partial z^2}\frac{\partial z}{\partial y} = \frac{\partial^2\varphi}{\partial z^2} \end{aligned} \tag{2}$$

Exercise 5: (0.5+1 points)

a) What is the volume of the tetrahedron T_1 with vertices

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}?$$

b) What is the volume of the tetrahedron T_2 with vertices

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 2 \end{bmatrix}?$$

Hint: there may be a transformation between T_2 and T_1.

Solution:

a) Using Fubini's theorem, the volume is:

$$\int_0^1\int_0^{1-z}\int_0^{1-z-y} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1\int_0^{1-z}(1-z-y)\mathrm{d}y\mathrm{d}z = \int_0^1 \left[(1-z)y - \frac{y^2}{2}\right]_0^{1-z} \mathrm{d}z$$
$$= \int_0^1 \frac{1}{2}(1-z)^2\mathrm{d}z = \left[-\frac{1}{6}(1-z)^3\right]_0^1 = \frac{1}{6}$$

b) The matrix $P = \begin{bmatrix} 2 & -1 & -2 \\ 1 & 3 & -5 \\ 1 & 1 & 2 \end{bmatrix}$ maps the tetrahedron T_1 onto the tetrahedron T_2. Using slide 8 of lecture 9 we have:

$$\mathrm{vol}\, T_2 = |\det P|\,\mathrm{vol}\, T_1 = \frac{33}{6} = \frac{11}{2}.$$

Exercise 6: (1.5 points) What is the area of the surface in $\mathbb{C}^3$ parametrized by $\gamma(z) = \begin{pmatrix} z^p \\ z^q \\ z^r \end{pmatrix}$, for $z \in \mathbb{C}, |z| \leq 1$?

Solution: We will use the formula of slide 13 of lecture 10. Let us call the surface S. It will be convenient to work in polar coordinates, so let $z = s(\cos\theta + \imath \sin\theta)$. Then, the surface S is equivalently parametrized by

$$\gamma(s,\theta) = \begin{bmatrix} s^p \cos p\theta \\ s^p \sin p\theta \\ s^q \cos q\theta \\ s^q \sin q\theta \\ s^r \cos r\theta \\ s^r \sin r\theta \end{bmatrix}.$$

It follows that

$$\mathrm{D}\gamma(s,\theta) = \begin{bmatrix} ps^{p-1}\cos p\theta & -ps^p \sin p\theta \\ ps^{p-1}\sin p\theta & ps^p \cos p\theta \\ qs^{q-1}\cos q\theta & -qs^q \sin q\theta \\ qs^{q-1}\sin q\theta & qs^q \cos q\theta \\ rs^{r-1}\cos r\theta & -rs^r \sin r\theta \\ rs^{r-1}\sin r\theta & rs^r \cos r\theta \end{bmatrix}.$$

Next we compute

$$\mathrm{D}\gamma(s,\theta)^\top \mathrm{D}\gamma(s,\theta) = \begin{bmatrix} p^2s^{2p-2} + q^2s^{2q-2} + r^2s^{2r-2} & 0 \\ 0 & p^2s^{2p} + q^2s^{2q} + r^2s^{2r} \end{bmatrix},$$

which leads to $\det\left(\mathrm{D}\gamma(s,\theta)^\top \mathrm{D}\gamma(s,\theta)\right) = \left(p^2s^{2p-1} + q^2s^{2q-1} + r^2s^{2r-1}\right)^2$. Finally we can compute:

$$\begin{aligned} \mathrm{vol}_k\, S &= \int_{|z|\leq 1} \sqrt{\det\left(\mathrm{D}\gamma(s,\theta)^\top \mathrm{D}\gamma(s,\theta)\right)}\, \mathrm{d}s\mathrm{d}\theta \\ &= \int_0^{2\pi}\int_0^1 \left(p^2s^{2p-1} + q^2s^{2q-1} + r^2s^{2r-1}\right) \mathrm{d}s\mathrm{d}\theta \\ &= \pi(p+q+r). \end{aligned}$$

Exercise 7: (1 point) Let $f : [a,b] \to \mathbb{R}$ be a smooth positive function. Find a parametrization for the surface of equation $\frac{x^2}{A^2} + \frac{y^2}{B^2} = (f(z))^2$.

Solution: A possible parametrization is $\gamma : (z,\theta) \mapsto (Af(z)\cos\theta, Bf(z)\sin\theta, z) = (x,y,z)$ with $z \in [a,b]$ and $\theta \in [0,2\pi]$.

Exercise 8: (1.5 points) Compute the following integral:

$$\int_{[\gamma(U)]} \sin y^2 \mathrm{d}x \wedge \mathrm{d}z, \text{ where } U = [0,a]\times[0,b], \text{ and } \gamma\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^2 - v \\ uv \\ v^4 \end{pmatrix}.$$

Solution: Since $\mathrm{D}\gamma = \begin{bmatrix} 2u & -1 \\ v & u \\ 0 & 4v^3 \end{bmatrix}$, we have:

$$\begin{aligned}
\int_{[\gamma(U)]} \sin y^2 \mathrm{d}x \wedge \mathrm{d}z &= \int_0^a \int_0^b \sin(u^2v^2)(8uv^3)\mathrm{d}u\mathrm{d}v \\
&= \int_0^b \left(\int_0^{a^2} \sin(xv^2)(4v^3)\mathrm{d}x \right) \mathrm{d}v \qquad \leftarrow x = u^2 \\
&= -\int_0^b 4v\cos(xv^2)\big|_0^{a^2} \mathrm{d}v = -\int_0^b 4v(\cos(a^2v^2) - 1)\mathrm{d}v \\
&= -\int_0^{a^2b^2} \frac{2}{a^2}\cos y \mathrm{d}y + \int_0^b 4v\mathrm{d}v \qquad \leftarrow y = a^2v^2 \\
&= 2b^2 - \frac{2}{a^2}\sin(a^2b^2).
\end{aligned} \tag{3}$$

Bonus questions

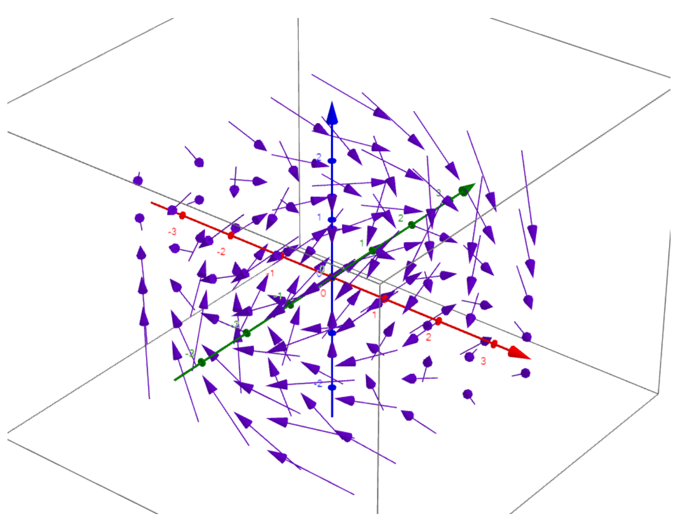
Exercise 9: (1 point) For the 1-form $y\mathrm{d}x - x\mathrm{d}y - z\mathrm{d}z$ on $\mathbb{R}^3$:

a) Write down the corresponding vector field
b) Sketch the vector field
c) Describe a path over which the work of the 1-form would be small
d) Describe a path over which the work would be large

Solution: (a) Note that the 1-form corresponds to the work of the vector field

$$\vec{F} = \begin{bmatrix} y \\ -x \\ -z \end{bmatrix}.$$

(b)



(c) Note that if we consider a path close to $\gamma(t) = (t, t, 0)$, the work will be small.

(d) The work will be large over a path of the form $\gamma(t) = (t, -t, -t)$.

Exercise 10: (1 point) Let $\vec{F}$ be the vector field $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \\ 0 \end{bmatrix}$, where F_1 and F_2 are defined on all of $\mathbb{R}^2$.

Suppose $\mathrm{D}_2F_1 = \mathrm{D}_1F_2$. Show that there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = \nabla f$

Solution: Simple application of Poincaré Lemma, see second item of slide 14 of lecture 15. For full points, the solution needs to be fully detailed by showing how and why $\mathrm{curl}\vec{F} = 0$, and mentioning how is it that Poincaré lemma leads to the answer.

Exercise 11: (1 point) Find the unique polynomial p such that $p(1) = 1$ and such that if

$$\omega = x\mathrm{d}y \wedge \mathrm{d}z - 2zp(y)\mathrm{d}x \wedge \mathrm{d}y + yp(y)\mathrm{d}z \wedge \mathrm{d}x$$

then $\mathbf{d}\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. For such polynomial p, find the integral $\int_S \omega$, where S is that part of the sphere $x^2+y^2+z^2 = 1$ where $z \geq \sqrt{2}/2$, oriented by the outward-pointing normal.

Solution:

$$\begin{aligned} \mathbf{d}\omega &= \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - 2p(y)\mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y + (p(y) + yp'(y))\mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}x \\ &= (1 - p(y) + yp'(y))\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \end{aligned} \tag{4}$$

So, we have to solve the differential equation $yp'(y) = p(y)$ which has general solution $p(y) = y + c$, but since we want $p(1) = 1$, then $p(y) = y$. In this way

$$\omega = x\mathrm{d}y \wedge \mathrm{d}z - 2yz\mathrm{d}x \wedge \mathrm{d}y + y^2\mathrm{d}z \wedge \mathrm{d}x, \tag{5}$$

and indeed $\mathbf{d}\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

Next, instead of computing $\int_S \omega$ (which would be very difficult) we notice that $\mathbf{d}\omega$ is a constant form and rather use Stokes theorem ($\int_S \omega = \int_M \mathbf{d}\omega$ where $S = \partial M$). We then see that S is the boundary of a piece of ball M. Using polar coordinates we have that M is parametrized by $\gamma(\theta, z) = (\underbrace{\sqrt{1-z^2}}_{r}\cos\theta, \underbrace{\sqrt{1-z^2}}_{r}\sin\theta, z)$ with $U = \{(\theta, z) \,|\, \theta \in [0, 2\pi],\, z \in [z_0, 1]\}$. Here we have chosen z_0 for generality, later we can substitute $z_0 = \frac{\sqrt{2}}{2}$. Notice that, geometrically with $\int_M \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, we are computing the volume of a cap of a sphere that starts at $z = z_0$. The integral is set up as follows:

$$\int_S \omega = \int_M \mathbf{d}\omega = \int_{z_0}^1 \int_0^{\sqrt{1-z^2}} \int_0^{2\pi} r\,\mathrm{d}\theta\mathrm{d}r\mathrm{d}z, \tag{6}$$

which simply tells us that we are adding, along $z \in [z_0, 1]$, the areas of circles of radius $\sqrt{1-z^2}$. So, we have:

$$\begin{aligned} \int_S \omega = \int_M \mathbf{d}\omega &= \int_{z_0}^1 \int_0^{\sqrt{1-z^2}} \int_0^{2\pi} r\,\mathrm{d}\theta\mathrm{d}r\mathrm{d}z \\ &= 2\pi \int_{z_0}^1 \int_0^{\sqrt{1-z^2}} r\,\mathrm{d}r\mathrm{d}z \\ &= \pi \int_{z_0}^1 (1-z^2)\mathrm{d}z = \pi(z - \frac{z^3}{3})\Big|_{z_0}^1 = \pi\left(\frac{2}{3} - z_0 + \frac{z_0^3}{3}\right) =: V(z_0). \end{aligned} \tag{7}$$

So, $V\left(\frac{\sqrt{2}}{2}\right) = \pi\left(\frac{2}{3} - \frac{1}{5\sqrt{2}}\right)$. As a side note $V(0) = \frac{2}{3}\pi = \frac{V(-1)}{2}$ as you would expect.